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LATTICE CONSTANT OF HYDRODYNAMIC DISSIPATIVE STRUCTURE

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The lattice constant of two dimentional hexagonal lattice appeared in the hydrodynamic dissipative structure of liquid layer is calculated from a viewpoint of thermodynamics. This calculation method is a rather general one which should be available for the Benard problem and the Felici instability. The axial ratio of the convective unit cell, γ, is calculated to be $\gamma=3^{1/4}/2 = 0.658$.

It is well known that when a liquid layer is heated below[1] or electrical charge is injected into the layer from electrode,[2] a convective motion takes place in the case where difference in temperature or electrical potential between the upper and the lower surface of the layer becomes larger than a critical value. It is also known that a dissipative structure appears in a suspension of photoconductive particles under illumination in an electric field.[3] These convective motions usually have two dimentional hexagonal symmetry, and the lattice constant of this structure has been known to be of the order of the thickness of the liquid layer. In the present letter we analize this convective structure from a viewpoint of thermodynamics.

Figure 1 shows a two dimensional hexagonal lattice with the lattice constant, a, and the Wigner-Seitz unit cell, ABCDEF. The point O is a lattice point of the hexagonal lattice, while the points A,B,C,D,E and F etc. compose the dual lattice of the hexagonal lattice. If the hexagonal lattice corresponds to the ascending flow of the fluid, then its dual lattice corresponds to the descending flow. In the unit cell ABCDEF the fluid flows up at the lattice point O, spreading toward the six directions and falls downward at the boundary of this unit cell, i.e. this flow is totally symmetric within this call.

Concerning the velocity of the liquid, $\mathbf{v}(\mathbf{r})$, we assume that the z-component, $v_z(\mathbf{r})$, can be separated into the horizontal(x,y) component and the vertical(z) component[4] as

$$V_z(x,y,z) = v\, f(x,y)\, g(z), \qquad (1)$$

where the z-direction is perpendicular to the liquid surface. Then the function $f(x,y)$ should satisfy the periodical condition of the hexagonal lattice with the lattice constant, a, and the function $g(z)$ should satisfy the boundary condition at the surfaces(z=0 and c). As the components of larger wave number in $f(x,y)$ and $g(z)$ are generally unfavable for a stationary state because they will increase the energy dissipation,[5] we take terms of the smallest wave number into consideration. Thus we obtain

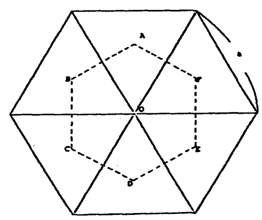

Fig.1 Hexagonal lattice and the Wigner-Seitz unit cell.

$$f(\mathbf{r}_1) = (1/6)\sum_{i=1}^{6} \exp(i\mathbf{g}_i \mathbf{r}_1),\quad g(z) = (1/2)[1-\cos(c^*z)], \qquad (2)$$

here $\mathbf{r}_1 \equiv (x,y)$, $c^* \equiv 2\pi/c$, and $\mathbf{g}_i$ (i=1,2,3,4,5,6) are the shortest reciprocal vectors of the hexagonal lattice with the length of $g=4\pi/(\sqrt{3}\ a)$. From Eq.(2) we have

$$(\nabla_1^2+(4/3)a^{*2})f=0, \quad (D^2+c^{*2})g=(1/2)c^{*2}, \quad (3)$$

here $\nabla_1 \equiv (\partial/\partial x, \partial/\partial y)$, $D \equiv \partial/\partial z$ and $a^* \equiv 2\pi/a$. With the continuity condition of incompressible fluid flow, div $\mathbf{v}$=0, the stream field is obtained as

$$v_x(\mathbf{r})=(3/4)\nu a^{*-2}(\partial f/\partial x)(dg/dz),$$
$$v_y(\mathbf{r})=(3/4)\nu a^{*-2}(\partial f/\partial y)(dg/dz), \quad (4)$$
$$v_z(\mathbf{r})=\nu fg.$$

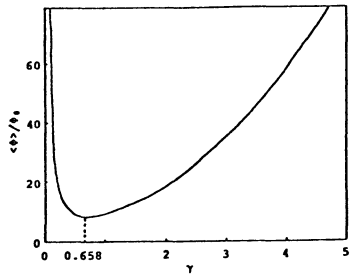

Fig.2 The mean viscous dissipative function

The amount of heat generated by the convection of Eq.(4) in a unit volume and unit time is given by the viscous dissipation function,

$$\Phi(\mathbf{r})=(1/2)\eta\,\Sigma\,(\partial v_i/\partial x_j+\partial v_j/\partial v_i)^2$$
$$=\eta\nu^2[\{4f^2+(9/4)a^{*-2}((\partial^2 f/\partial x\partial y)^2-(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2))\}(dg/dz)^2$$
$$+\{(\partial f/\partial x)^2+(\partial f/\partial y)^2\}\{g+(3/4)a^{*-2}(d^2g/dz^2)\}^2]. \quad (5)$$

Integrating this over a unit cell, then dividing the result by the unit cell volume, $(\sqrt{3}/2)a^2c$, the mean viscous dissipation, $\langle\Phi\rangle$, is obtained as

$$\langle\Phi\rangle=\int_{\text{unit cell}}\Phi(x,y,z)dV/\int_{\text{unit cell}}dV$$
$$=\Phi_0\pi^2[(1/3)\gamma^2+(1/16)\gamma^{-2}+(1/2)], \quad (6)$$

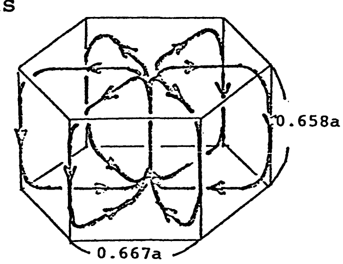

Fig.3 The unit cell of the convection

here $\gamma \equiv c/a$ is the axial ratio of this convection and $\Phi_0=\eta\nu^2/c^2$. Figure 2 shows the variation of $\langle\Phi\rangle$ with the virtual variation of the axial ratio. This curve has a minimum at $\gamma=3^{1/4}\times 2^{-1}=0.658$ (Figure 3). Assuming that the principle of the minimum energy dissipation is applicable in the present condition, we conclude that this convective structure has the hexagonal lattice with the axial ratio of γ=0.658. This means that the lattice constant, a, is proportional to the liquid layer thickness, c as

$$a=1.520\ c. \quad (7)$$

It has been observed that the lattice constant of the dissipative structure is proportional to and somewhat larger than the layer thickness.[1-4] Our theoretical result is in good agreement with those experimental results.

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